

5-DIMENSIONAL CONTACT $\mathrm{SO}(3)$ -MANIFOLDS AND DEHN TWISTS

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ABSTRACT. In this paper the 5-dimensional contact $\mathrm{SO}(3)$ -manifolds are classified up to equivariant contactomorphisms. The construction of such manifolds with singular orbits requires the use of generalized Dehn twists.

We show as an application that all simply connected 5-manifolds with singular orbits are realized by a Brieskorn manifold with exponents $(k, 2, 2, 2)$. The standard contact structure on such a manifold gives right-handed Dehn twists, and a second contact structure defined in the article gives left-handed twists.

A 5-dimensional contact $\mathrm{SO}(3)$ -manifold M can be decomposed into the set of singular orbits $M_{(\mathrm{sing})}$ and the set of regular orbits $M_{(\mathrm{reg})}$. Both parts can be described relatively easily: The singular orbits are the disjoint union of copies of $\mathbb{S}^1 \times \mathbb{RP}^2$, $\mathbb{S}^1 \times \mathbb{S}^2$ or $\mathbb{S}^1 \widetilde{\times} \mathbb{S}^2 := \mathbb{R} \times \mathbb{S}^2 / \sim$, where $(t, p) \sim (t + 1, -p)$. The set of regular orbits contains a canonical submanifold R of dimension 3 (the so-called cross-section), and one has that $M_{(\mathrm{reg})} \cong \mathrm{SO}(3) \times_{\mathbb{S}^1} R$.

For gluing the singular orbits onto the regular ones, there is an integer invariant that classifies all possibilities. This integer corresponds to the number of Dehn twists.

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0. NOTATION

This section only fixes some notations about Lie groups and G -manifolds. In the article, G denotes always a compact, connected Lie group, \mathfrak{g} is its Lie algebra and \mathfrak{g}^* is the corresponding coalgebra. The only G -operation considered on \mathfrak{g}^* will be the coadjoint action. For the stabilizer of an element $\nu \in \mathfrak{g}^*$, we write G_ν .

A G -equivariant map Φ between G -manifolds M and N consists of a smooth map $\Phi_M : M \rightarrow N$ such that $\Phi_M(gp) = g\Phi_M(p)$. As a short-hand, we will write G -diffeomorphism instead of G -equivariant diffeomorphism, G -contactomorphism instead of G -equivariant contactomorphism etc.

Let N be a submanifold of a G -manifold M . The **flow-out** of N is defined as the set $G \cdot N$.

For a G -manifold M , we denote the set of principal orbits by $M_{(\mathrm{princ})}$, the set of singular orbits by $M_{(\mathrm{sing})}$ and the set of regular (i.e. non-singular) orbits by $M_{(\mathrm{reg})}$. The conjugation class of a closed subgroup $H \leq G$ is written (H) , and $M_{(H)}$ is the set of points $p \in M$ whose stabilizer $\mathrm{Stab}(p)$ lies in the class (H) . The normalizer $N(H)$ of H is the subgroup $\{g \in G \mid gHg^{-1} = H\}$.

For every element $X \in \mathfrak{g}$, the **infinitesimal generator** of the action is the vector field $X_M(p) := \left. \frac{d}{dt} \right|_{t=0} \exp(tX)p$.

1. PRELIMINARIES

At any point p of a G -manifold, there exists a so-called **slice** S_p . This is a submanifold that is transverse to the orbit $\mathrm{Orb}(p)$, invariant under the action of $\mathrm{Stab}(p)$, and satisfies the condition that whenever $g \cdot q \in S_p$ (with $g \in G$ and $q \in S_p$), then $g \in \mathrm{Stab}(p)$. In particular for the coadjoint action on \mathfrak{g}^* , there exists a unique maximal slice at any $\nu \in \mathfrak{g}^*$, which will be denoted by S_ν^* (see [DK00]).

Example 1. Consider the $\mathrm{SO}(3)$ -structure of $\mathfrak{so}(3)^*$ given by the coadjoint action. The principal orbits are 2-spheres lying concentrically around 0, and $\{0\}$ is the only singular orbit in $\mathfrak{so}(3)^*$. The maximal slice of an element $\nu \in \mathfrak{so}(3)^*$ ($\nu \neq 0$) is $\mathbb{R}^+ \cdot \nu$ and the maximal slice at 0 is the whole of $\mathfrak{so}(3)^*$.

Definition. A **contact G -manifold** (M, α) is a G -manifold with an invariant contact form α .¹

In the rest of the article we will assume that all contactomorphisms **preserve the coorientation** of the contact structure, i.e. for a contactomorphism $\Phi : (M, \alpha) \rightarrow (M', \alpha')$ with $\Phi^* \alpha' = f \alpha$, the function f has to be positive.

Definition. The **moment map** $\mu : M \rightarrow \mathfrak{g}^*$ of a contact G -manifold (M, α) is given by

$$\langle \mu(p) | X \rangle := \alpha_p(X_M) .$$

Definition. For a contact G -manifold (M, α) with moment map $\mu : M \rightarrow \mathfrak{g}^*$, the **cross-section R at a point $\nu \in \mu(M)$** is defined as

$$R := \mu^{-1}(S_\nu^*) .$$

One can find a symplectic version of the following theorem in [LMTW98], the contact version has been described in [Wil02].

Theorem 1 (cross-section theorem). *Let (M, α) be a contact G -manifold with moment map $\mu_M : M \rightarrow \mathfrak{g}^*$. Let $\nu \in \mathfrak{g}^*$ be an element in the image of the moment map, and let $S_\nu^* \subseteq \mathfrak{g}^*$ be the unique maximal slice at ν .*

Then:

- (1) *The **cross-section** $R := \mu_M^{-1}(S_\nu^*)$ is a contact G_ν -submanifold of M , where $G_\nu := \mathrm{Stab}(\nu)$.*
- (2) *The G -action induces a G -diffeomorphism between the flow-out $G \cdot R \subseteq M$ and $G \times_{G_\nu} R$. The contact form α on the flow-out can be reconstructed from the cross-section and the embedding $\iota : G_\nu \hookrightarrow G$.*

Remark 1. Note that the action of G_ν on the cross-section is in general not effective (even if the G -action on M was).

Remark 2. The theorem uses the embedding $G_\nu \hookrightarrow G$. If one considers a cross-section R as an abstract H -manifold with $H \cong G_\nu$ and one embeds H in two different ways into G ($\iota_1, \iota_2 : H \hookrightarrow G$), then in general $G \times_{\iota_1 H} R \not\cong G \times_{\iota_2 H} R$. In the case of $\mathrm{SO}(3)$ -manifolds however, the embedding of \mathbb{S}^1 into $\mathrm{SO}(3)$ is unique up to conjugation, and no problem will arise at this point.

In the following corollary, the cross-section theorem will be applied to 5-dimensional contact $\mathrm{SO}(3)$ -manifolds.

Corollary 2. *Let (M, α) be a 5-dimensional contact $\mathrm{SO}(3)$ -manifold with moment map $\mu : M \rightarrow \mathfrak{so}(3)^*$. The cross-section R is a 3-dimensional contact \mathbb{S}^1 -manifold without Legendrian orbits or fixed points.*

Conversely, let (R, α) be a 3-dimensional contact \mathbb{S}^1 -manifold without Legendrian orbits, and fixed points. Then there is a 5-dimensional contact $\mathrm{SO}(3)$ -manifold M that has R as its cross-section.

Proof. The first part of the statement is a direct consequence of the cross-section theorem and Example 1. If R had Legendrian orbits or fixed points, then 0 would be contained in the image $\mu(R)$.

For the second part, the manifold M is given by $\mathrm{SO}(3) \times_{\mathbb{S}^1} R$, with the standard $\mathrm{SO}(3)$ -action on the left factor. The contact form on M is constructed by taking $\alpha + \alpha(Z_R) \cdot Z^*$ on $\{e\} \times_{\mathbb{S}^1} R$, and moving it with the $\mathrm{SO}(3)$ -action to the rest of M . With Z^* , we mean the dual of Z with respect to the standard basis $\{X, Y, Z\}$ of $\mathfrak{so}(3)$. \square

¹If $\xi = \ker \alpha$ is a G -invariant contact structure on M , then one can average α over the G -action to obtain an invariant contact form.

Lemma 3. *Let (M, α) and (M', α') be 5-dimensional contact $\mathrm{SO}(3)$ -manifolds. An $\mathrm{SO}(3)$ -contactomorphism $\Phi : M \rightarrow M'$ induces an \mathbb{S}^1 -contactomorphism between the cross-sections R and R' .*

Proof. The pull-back $\Phi^* \alpha'$ is equal to $f \alpha$ with a positive function $f : M \rightarrow \mathbb{R}$. For the moment maps, this gives $\mu' \circ \Phi = f \cdot \mu$. The restriction of Φ to R is then an \mathbb{S}^1 -contactomorphism to R' . \square

Lemma 4. *Let (M, α) and (M', α') be 5-dimensional contact $\mathrm{SO}(3)$ -manifolds, and let R and R' be their respective cross-sections. An \mathbb{S}^1 -contactomorphism $\Phi : R \rightarrow R'$ induces an $\mathrm{SO}(3)$ -contactomorphism between the flow-outs $\mathrm{SO}(3) \cdot R \subset M$ and $\mathrm{SO}(3) \cdot R' \subset M'$.*

Proof. The map is given by

$$\mathrm{SO}(3) \times_{\mathbb{S}^1} R \rightarrow \mathrm{SO}(3) \times_{\mathbb{S}^1} R', \quad [g, p] \mapsto [g, \Phi(p)].$$

One easily checks that these maps are well-defined, and respect the contact structures. \square

2. 5-DIMENSIONAL CONTACT $\mathrm{SO}(3)$ -MANIFOLDS

The classification of closed symplectic 4-manifolds with a Hamiltonian $\mathrm{SO}(3)$ - or $\mathrm{SU}(2)$ -action was given in [Igl91] and [Aud91]. In the rest of the article, a proof to the theorem below will be given, which describes the classification of 5-dimensional contact $\mathrm{SO}(3)$ -manifolds.

Theorem 5. *The following list gives a complete set of invariants for cooriented 5-dimensional closed contact $\mathrm{SO}(3)$ -manifold M , in the sense that there is an $\mathrm{SO}(3)$ -contactomorphism between any two manifolds with equal invariants, and there exists a manifold for every choice of invariants from the list.*

- The principal stabilizer is isomorphic to \mathbb{Z}_k for some $k \in \mathbb{N}$ (including the trivial group, for $k = 1$).
- The closure \overline{R} of the cross-section is a compact 3-dimensional contact \mathbb{S}^1 -manifold without any fixed points or special exceptional orbits. Each boundary component of \overline{R} corresponds to a component of $M_{(\mathrm{sing})}$. The orbits in the boundary are the only Legendrian orbits.
- If M has singular orbits, then the principal stabilizer is either isomorphic to \mathbb{Z}_2 or trivial. In the first case, all components of $M_{(\mathrm{sing})}$ are isomorphic to $\mathbb{S}^1 \times \mathbb{RP}^2$. If the principal stabilizer is trivial, one has two different types of components in $M_{(\mathrm{sing})}$, which are either copies of $\mathbb{S}^1 \times \mathbb{S}^2$ or $\mathbb{S}^1 \widetilde{\times} \mathbb{S}^2 := \mathbb{R} \times \mathbb{S}^2 / \sim$ with the equivalence $(t, p) \sim (t + 1, -p)$. The Dehn-Euler number $n(R)$ is an integer, which describes how $M_{(\mathrm{sing})}$ is glued onto $M_{(\mathrm{reg})}$. This Dehn-Euler number satisfies certain arithmetic conditions described in the Definition on page 10.

Remark 3. Contact 3-dimensional \mathbb{S}^1 -manifolds have been classified in [KT91]. The cross-section R is thus determined by the following invariants:

- If R is closed, it is determined solely by the genus of its orbit space $B := R/\mathbb{S}^1$, the exceptional orbits, and the orbifold Euler number which cannot be zero.
- If R is an open manifold, it is determined by the number of boundary components, the genus of its orbit space B , and its exceptional orbits.

Let (M, α) be a contact 5-manifold and let $\mathrm{SO}(3)$ act by contact transformations with moment map μ .

Lemma 6. *The principal stabilizer of a contact $\mathrm{SO}(3)$ -manifold is isomorphic to \mathbb{Z}_k for some $k \in \mathbb{N}$ (including the trivial group, for $k = 1$).*

Proof. Since the moment map μ corresponding to the action is equivariant, $\mathrm{Stab}(p) \leq \mu(\mathrm{Stab}(p))$. The $\mathrm{SO}(3)$ -structure of $\mathfrak{so}(3)^*$ was given in Example 1, and it follows that $\mu \equiv 0$ if the principal stabilizer is not one of \mathbb{Z}_k or \mathbb{S}^1 . But $\mu \equiv 0$ means that the action is trivial, which in particular contradicts effectiveness.

In fact, the circle \mathbb{S}^1 can also be excluded: Assume $\exp(tX)$ (for some $X \in \mathfrak{so}(3)$, $X \neq 0$) leaves p fixed, i.e. $\exp(tX) \cdot p = p$, then we have $\mu(p) = \mu(\exp(tX) \cdot p) = \mathrm{Ad}(\exp(-tX))^* \mu(p)$

and as a consequence $\text{ad}(X)^*\mu(p) = 0$. Let now $X, Y, Z \in \mathfrak{so}(3)$ be a standard basis of the Lie algebra. Then, $\langle \mu(p)|Z \rangle = \langle \mu(p)|[X, Y] \rangle = 0$, $\langle \mu(p)|Y \rangle = -\langle \mu(p)|[X, Z] \rangle = 0$ and obviously $\langle \mu(p)|X \rangle = \alpha(X_M(p)) = 0$, i.e. $\mu(p) = 0$.

Not only does this show that \mathbb{S}^1 cannot be a principal stabilizer, it also proves that all singular orbits lie in $\mu^{-1}(0)$, and the cross-section has no fixed points. \square

The principal cross-section $R = \mu^{-1}(\mathbb{R}^+ Z^*)$ is a contact 3-manifold with a Hamiltonian \mathbb{S}^1 -action. The \mathbb{S}^1 -orbits are neither fixed points nor tangent to the contact structure. If $0 \notin \mu(M)$ the cross-section R is a closed subset of M , because $\mathbb{R}^+ Z^* \cap \mu(M)$ is compact, and hence R is a closed manifold and then M , as flow-out of R , is completely determined by R .

Lemma 7. *Let (M, α) be a 5-dimensional contact $\text{SO}(3)$ -manifold. Then $M_{(\text{sing})} = \mu^{-1}(0)$.*

Proof. The preimage $\mu^{-1}(0)$ is the union of $\text{SO}(3)$ -orbits tangent to $\ker \alpha$, i.e. a collection of isotropic submanifolds. But isotropic submanifolds of a 5-dimensional contact manifold have at most dimension 2, and hence these orbits have to be singular. On the other hand, the proof of Lemma 6 shows that all singular orbits lie in $\mu^{-1}(0)$. \square

Furthermore a stabilizer of an exceptional orbit is isomorphic to some \mathbb{Z}_m and these orbits lie discrete surrounded by principal orbits.

2.1. Examples. In this section a few examples will be introduced that are continued later in the article, while the theory is developed.

Example 2. The standard contact structure on the 5-sphere $\mathbb{S}^5 \subset \mathbb{C}^3$ is given at a point (z_1, z_2, z_3) by

$$\alpha_+ = \sum_{j=1}^3 (x_j dy_j - y_j dx_j),$$

with $z_j = x_j + iy_j$. This contact form is invariant under the $\text{SO}(3)$ -action induced by the standard matrix representation.

The stabilizer of a point $\mathbf{x} + i\mathbf{y} \in \mathbb{S}^5$ with $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ is the intersection of the stabilizer of \mathbf{x} and that of \mathbf{y} . If \mathbf{x} and \mathbf{y} are linearly independent, we have $\text{Stab}(\mathbf{x} + i\mathbf{y}) = \{e\}$ and $\text{Stab}(\mathbf{x} + i\mathbf{y}) \cong \mathbb{S}^1$ otherwise.

For any matrix $A \in \mathfrak{so}(3)$, the moment map is given by $\langle \mu(\mathbf{x} + i\mathbf{y})|A \rangle = 2\mathbf{x}^t A \mathbf{y}$. The cross-section is then the set

$$R = \{\mathbf{x} + i\mathbf{y} \in \mathbb{S}^5 | x_1 y_3 - y_1 x_3 = x_2 y_3 - y_2 x_3 = 0 \text{ and } x_1 y_2 - y_1 x_2 > 0\}.$$

The condition $x_1 y_2 - y_1 x_2 > 0$ implies that the other two equations, regarded as a linear system in (x_3, y_3) , have the unique solution $(x_3, y_3) = 0$. Hence the cross-section is given by

$$R = \{(z_1, z_2, 0) \in \mathbb{S}^5 | x_1 y_2 - y_1 x_2 > 0\}.$$

The \mathbb{S}^1 -action on R is given by simultaneous rotations in the (x_1, x_2) - and (y_1, y_2) -plane. Its orbit space R/\mathbb{S}^1 lies in a natural way in \mathbb{CP}^1 with the projection $\pi : R \rightarrow R/\mathbb{S}^1$ given by $\pi(x_1 + iy_1, x_2 + iy_2, 0) = [x_1 + ix_2 : y_1 + iy_2]$. Note that the equation $x_1 y_2 - x_2 y_1 = 0$ is well-defined in \mathbb{CP}^1 and its solutions are given by the standard embedding of \mathbb{RP}^1 . Hence R/\mathbb{S}^1 is diffeomorphic to an open disc and $R \cong \mathbb{D}_{<1}^2 \times \mathbb{S}^1$.

Another $\text{SO}(3)$ -invariant contact form on \mathbb{S}^5 can be given by

$$\begin{aligned} \alpha_- = i \sum_{j=1}^3 (z_j d\bar{z}_j - \bar{z}_j dz_j) \\ - i((z_1^2 + z_2^2 + z_3^2) d(\bar{z}_1^2 + \bar{z}_2^2 + \bar{z}_3^2) - (\bar{z}_1^2 + \bar{z}_2^2 + \bar{z}_3^2) d(z_1^2 + z_2^2 + z_3^2)). \end{aligned}$$

Note that the first part of the form is identical to the standard form α_+ . It is easy to check that the second term does not give any contribution to the moment map, and hence $\mu_+ = \mu_-$. The cross-section for α_+ and α_- are then of course also equal.

The example will be continued at the end of the next section.

I would like to thank Otto van Koert for pointing out the following examples to me. As we will see later, these are all the simply connected contact SO(3)-manifolds with singular orbits of dimension 5. A good reference is [HM68] and [LM76]. The open book decomposition of these examples is closely related to the SO(3)-symmetry ([vKN]).

Example 3. The Brieskorn manifolds $W_k^5 \subset \mathbb{C}^4$ (with $k \in \mathbb{N}_0$) are defined as the intersection of the 7-sphere with the zero set of the polynomial $f(z_0, z_1, z_2, z_3) = z_0^k + z_1^2 + z_2^2 + z_3^2$. To make computations easier, assume the radius of the 7-sphere to be $\sqrt{2}$. It is well-known that W_k^5 is diffeomorphic to \mathbb{S}^5 for k odd, and to $\mathbb{S}^2 \times \mathbb{S}^3$ for k even.

Let SO(3) act linearly on \mathbb{C}^4 by leaving the first coordinate of (z_0, z_1, z_2, z_3) fixed and multiplying the last three coordinates with SO(3) in its real standard representation, i.e. $A \cdot (z_0, z_1, z_2, z_3) := (z_0, A \cdot (z_1, z_2, z_3))$. It is easy to check that this action restricts to W_k^5 , because the polynomial f can be written as $z_0^k + \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2 + 2i\langle \mathbf{x} | \mathbf{y} \rangle$ with $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$. The only stabilizers that occur are $\{e\}$ and \mathbb{S}^1 . A point lies on a principal orbit, if and only if \mathbf{x} and \mathbf{y} are linearly independent.

Finally the invariant 1-forms

$$\alpha_k = (k+1) \cdot (x_0 dy_0 - y_0 dx_0) + 2 \sum_{j=1}^3 (x_j dy_j - y_j dx_j)$$

and

$$\alpha_{-k} = -(k+1) \cdot (x_0 dy_0 - y_0 dx_0) + 2 \sum_{j=1}^3 (x_j dy_j - y_j dx_j)$$

are both of contact type on W_k^5 .

The infinitesimal generators of the SO(3)-action do not have a z_0 -component. Hence the moment maps $\mu_k(z_0, z_1, z_2, z_3)$ for both α_k and α_{-k} are equal. They are given by

$$\langle \mu_k | X \rangle = 4(x_3 y_2 - x_2 y_3), \quad \langle \mu_k | Y \rangle = 4(x_1 y_3 - x_3 y_1), \quad \text{and} \quad \langle \mu_k | Z \rangle = 4(x_2 y_1 - x_1 y_2).$$

It can be seen with a similar computation as in Example 2 that the cross-section R is given by the points $(z_0, z_1, z_2, 0) \in W_k^5$ with $x_2 y_1 - x_1 y_2 > 0$.

The map $(z_0, z_1, z_2, 0) \mapsto z_0$ from R to the open unit disc is the projection of R onto its quotient space (see [HM68]). The cross-section is \mathbb{S}^1 -diffeomorphic to $\mathbb{D}_{<1}^2 \times \mathbb{S}^1$.

The example will be continued at the end of the next section.

2.2. Singular orbits. In this section, we will show that each component of $M_{(\text{sing})}$ corresponds to one of three possible models.

Lemma 8. *Let (M, α) be a 5-dimensional closed contact SO(3)-manifold. Recall from Lemma 6 that the principal stabilizer H is either trivial or isomorphic to \mathbb{Z}_k .*

If $H \cong \mathbb{Z}_k$ with $k \geq 3$, then M has no singular orbits.

If $H \cong \mathbb{Z}_2$, then any component of $M_{(\text{sing})}$ has a neighborhood that is SO(3)-diffeomorphic to a neighborhood of the zero-section in $\mathbb{S}^1 \times T\mathbb{R}P^2$, with trivial action on the first part and natural action on the second one.

If H is trivial, any component of $M_{(\text{sing})}$ has a neighborhood that is SO(3)-diffeomorphic to a neighborhood of the zero-section in the vertical bundle VE_{triv} or VE_{twist} , where E_{triv} is the trivial \mathbb{S}^2 -bundle over \mathbb{S}^1 and E_{twist} is the twisted \mathbb{S}^2 -bundle over \mathbb{S}^1 .

In all of these cases, there is up to SO(3)-contactomorphisms a unique invariant contact form on sufficiently small neighborhoods of $M_{(\text{sing})}$.

In the rest of this section we will describe all possible cases, and show the claims of the lemma.

One of the conclusion will be that the closure of the cross-section of a 5-dimensional contact SO(3)-manifold M is a compact 3-dimensional contact \mathbb{S}^1 -manifold with boundary. The interior points of R lie in regular SO(3)-orbits, while ∂R lies in $M_{(\text{sing})}$. The \mathbb{S}^1 -orbits at the boundary are Legendrian.

Lemma 9 (Equivariant Weinstein Theorem). *Let $\text{Orb}(p) \hookrightarrow M$ be a Legendrian $\text{SO}(3)$ -orbit. Then a neighborhood of $\text{Orb}(p)$ is $\text{SO}(3)$ -contactomorphic to a neighborhood of the zero-section in $(\mathbb{R} \oplus T^*\text{Orb}(p), dt + \lambda_{\text{can}})$, where $\text{SO}(3)$ acts by $g \cdot (t, v) = (t, g_*^{-1}v)$.*

Proof. There is an $\text{SO}(3)$ -invariant almost complex structure J on the contact structure $\xi = \ker \alpha$ such that

$$T_q \text{Orb}(p) \cap J \cdot (T_q \text{Orb}(p)) = \{0\} \text{ for all } q \in \text{Orb}(p).$$

The trivial line bundle ε^1 spanned by the Reeb vector field of α is also $\text{SO}(3)$ -invariant. This implies that the normal bundle of $T \text{Orb}(p)$ in M can be equivariantly identified with $\varepsilon^1 \oplus T \text{Orb}(p) \cong \varepsilon^1 \oplus T^* \text{Orb}(p)$. The contact form restricts to $dt + c \lambda_{\text{can}}$ on the zero-section, and rescaling the fibre gives the desired form $dt + \lambda_{\text{can}}$. This allows us to apply [LW01, Theorem 5.2], which states that there is a neighborhood of the orbit $\text{SO}(3)$ -contactomorphic to the normal bundle. \square

By looking at the different stabilizers that can occur, it will be seen that all singular orbits are either isomorphic to \mathbb{S}^2 with stabilizer \mathbb{S}^1 or to \mathbb{RP}^2 with stabilizer $\text{O}(2)$.

2.2.1. Fixed points. The irreducible representations of $\text{SO}(3)$ are all odd-dimensional. This implies that 5-dimensional contact $\text{SO}(3)$ -manifolds do not have fixed points by the following argument. The vector space spanned by the Reeb field is a trivial submodule of $T_p M$, and the contact plane (ξ_p, J_p) is a complex 2-dimensional $\text{SO}(3)$ -module, which also has to be trivial. That means the action on $T_p M$ is trivial, which contradicts effectiveness.

2.2.2. Stabilizer $\text{O}(2)$. The neighborhood of an orbit with stabilizer $\text{O}(2)$ is $\text{SO}(3)$ -equivariant to $\mathbb{R} \times T^* \text{Orb}(p)$ with $\text{Orb}(p) \cong \mathbb{RP}^2$. The stabilizer of any non-zero element in $T^* \mathbb{RP}^2$ is isomorphic to \mathbb{Z}_2 , which is then the principal stabilizer.

A connected component of $M_{(\text{O}(2))}$ is an \mathbb{RP}^2 -bundle over \mathbb{S}^1 (the closure $\overline{M_{(\text{O}(2))}}$ is a closed submanifold, possibly containing points with larger stabilizer than $\text{O}(2)$, but we proved that M has no fixed points, and hence $\overline{M_{(\text{O}(2))}} = M_{(\text{O}(2))}$). The structure group of a (G/H) -bundle with the standard G -action on the fibers are just the G -equivariant diffeomorphisms from G/H to itself. It is not very difficult to see that these are given by $N(H)/H$ (see [Bre72]). In our case $N(\text{O}(2))/\text{O}(2) = \text{O}(2)/\text{O}(2) = \{e\}$, and hence every component of $M_{(\text{O}(2))}$ is of the form $\mathbb{S}^1 \times \mathbb{RP}^2$. The neighborhood of such a component is $\text{SO}(3)$ -diffeomorphic to $\mathbb{S}^1 \times T^* \mathbb{RP}^2$ with the standard $\text{SO}(3)$ -action on the second part. A possible invariant contact form is given by $dt + \lambda_{\text{can}}$, where λ_{can} is the canonical 1-form on $T^* \mathbb{RP}^2$.

In fact, the contact form above is the only one in a small neighborhood of the singular orbit up to $\text{SO}(3)$ -contactomorphisms. This can be proved in a similar way as Lemma 9: After pulling back the form to $\mathbb{S}^1 \times T^* \mathbb{RP}^2$, one has $\alpha = f(t) dt + r(t) \lambda_{\text{can}}$ on the singular orbits. One can divide by $f(t)$ and then rescale the fibres to obtain the standard form $dt + \lambda_{\text{can}}$, which allows us to use again the Theorem from [LW01].

In 2.3 and 2.4, it will be important to know how the cross-section looks like in a neighborhood of the singular orbits. We compute the cross-section close to $M_{(\text{O}(2))}$ in a coordinate description.

A chart of \mathbb{RP}^2 around $[1 : 0 : 0]$ is given by $\mathbb{R}^2 \rightarrow \mathbb{RP}^2$, $(q_1, q_2) \mapsto [1 : q_1 : q_2]$, and the $\text{SO}(3)$ -action is induced by the standard matrix representation. Let X, Y, Z be the standard basis of $\mathfrak{so}(3)$, where each element generates the rotation around the corresponding axis of \mathbb{R}^3 . For Y , for example the action looks like

$$\begin{aligned} \exp(tY) \cdot [1 : q_1 : q_2] &= [\cos t + q_2 \sin t : q_1 : q_2 \cos t - \sin t] \\ &= \left[1 : \frac{q_1}{\cos t + q_2 \sin t} : \frac{q_2 \cos t - \sin t}{\cos t + q_2 \sin t} \right] \end{aligned}$$

The infinitesimal generators of the action are given in this chart by

$$\begin{aligned} X_{\mathbb{RP}^2}([1 : q_1 : q_2]) &= q_2 \partial_{q_1} - q_1 \partial_{q_2} , \\ Y_{\mathbb{RP}^2}([1 : q_1 : q_2]) &= -q_1 q_2 \partial_{q_1} - (1 + q_2^2) \partial_{q_2} , \\ Z_{\mathbb{RP}^2}([1 : q_1 : q_2]) &= -(1 + q_1^2) \partial_{q_1} - q_1 q_2 \partial_{q_2} . \end{aligned}$$

and the moment map is

$$\begin{aligned}\langle \mu(t, q_1, q_2, p_1, p_2) | X \rangle &= q_2 p_1 - q_1 p_2, \\ \langle \mu(t, q_1, q_2, p_1, p_2) | Y \rangle &= -q_1 q_2 p_1 - (1 + q_2^2) p_2, \\ \langle \mu(t, q_1, q_2, p_1, p_2) | Z \rangle &= -(1 + q_1^2) p_1 - q_1 q_2 p_2.\end{aligned}$$

Elements of $\mu^{-1}(\mathbb{R}^+ Z^*)$ have $p_1 \neq 0$ or $p_2 \neq 0$, and for such elements $q_2 p_1 - q_1 p_2 = 0$ and $-q_1 q_2 p_1 - (1 + q_2^2) p_2 = 0$ hold. These two equations can be read as a linear system in p_1 and p_2 , and there are only non-trivial solutions if the corresponding determinant vanishes, that is, if $-q_2(1 + q_2^2) - q_1^2 q_2 = -q_2(1 + q_1^2 + q_2^2) = 0$. If this is the case, then $q_2 = 0$, and from this it follows that $p_2 = 0$. The cross-section R consists of vectors in $T\mathbb{RP}^2$ tangent to \mathbb{RP}^1 , but pointing only in positive direction (with the embedding of \mathbb{RP}^1 in \mathbb{RP}^2 given by $[a : b] \mapsto [a : b : 0]$).

The restriction of the contact form on R is given in the chart above by $dt + p_1 dq_1$. Hence α is of contact type even on the boundary of \overline{R} , and the orbits of the \mathbb{S}^1 -action are Legendrian on $\partial \overline{R} \cong \mathbb{S}^1 \times \mathbb{S}^1$.

A collar neighborhood of $\partial \overline{R}$ is of the form $\mathbb{S}^1 \times [0, \varepsilon) \times \mathbb{S}^1$ with contact form $dt + r d\varphi$ and action $e^{i\vartheta} \cdot (t, r, \varphi) = (t, r, \varphi + 2\vartheta)$. The embedding of this neighborhood into M is given by

$$(t, r, \varphi) \mapsto (t, [\cos(\varphi/2) : \sin(\varphi/2) : 0], -r \sin(\varphi/2) \partial_1 + r \cos(\varphi/2) \partial_2),$$

and the points $(t, 0, 0) \in \partial \overline{R}$ all have equal stabilizer in $\mathrm{SO}(3)$.

2.2.3. Stabilizer \mathbb{S}^1 . The neighborhood of such an orbit is $\mathrm{SO}(3)$ -diffeomorphic to $\mathbb{R} \times T\mathbb{S}^2$ with trivial action on the first and standard action on the second component. The principal stabilizer is trivial. A connected component of $M_{(\mathrm{SO}(2))}$ is a closed manifold, because no fixed points or points with stabilizer $\mathrm{O}(2)$ do exist, and hence $M_{(\mathrm{SO}(2))}$ is diffeomorphic to an \mathbb{S}^2 -bundle over \mathbb{S}^1 . The structure group of such a bundle is $N(\mathrm{SO}(2))/\mathrm{SO}(2) \cong \mathbb{Z}_2$, hence the only two \mathbb{S}^2 -bundles over \mathbb{S}^1 are the trivial one E_{triv} and the twisted one E_{twist} . They can be described by the equivalence relations $(t, p) \sim (t + 1, p)$ and $(t, p) \sim (t + 1, -p)$ (with $t \in \mathbb{R}$ and $p \in \mathbb{S}^2$) respectively. A neighborhood of a component of $M_{(\mathrm{sing})}$ is diffeomorphic to the corresponding vertical bundle. The $\mathrm{SO}(3)$ -action on the second component of $\mathbb{R} \times \mathbb{S}^2$ is compatible with these identifications, and one obtains an action on either vertical bundle VE_{triv} and VE_{twist} .

A possible invariant contact form is given by $dt + \lambda_{\mathrm{can}}$ on $\mathbb{R} \times T^*\mathbb{S}^2$, where $T^*\mathbb{S}^2$ is identified with $T\mathbb{S}^2$ via an invariant metric. This form descends to VE_{triv} and also to VE_{twist} , because the reflection in the construction of E_{twist} is induced by a diffeomorphism of \mathbb{S}^2 , and λ_{can} on T^*N remains invariant under maps induced by diffeomorphisms of the base space N .

In a small neighborhood of $M_{(\mathrm{SO}(2))}$, every invariant contact form is $\mathrm{SO}(3)$ -contactomorphic to $dt + \lambda_{\mathrm{can}}$. The proof of this fact is completely analogous to the one for orbits with stabilizer $\mathrm{O}(2)$ above, and will be omitted.

Now we will describe how the cross-section looks like in a neighborhood of the singular orbits. The moment map μ is given in the neighborhood of a singular orbit by

$$\langle \mu(t, q, p) | X \rangle = p^t X q$$

with $(t, q, p) \in \mathbb{R}^1 \times T^*\mathbb{S}^2 \subseteq \mathbb{R}^1 \times \mathbb{R}^3 \times \mathbb{R}^3$ and $X \in \mathfrak{so}(3)$ in its standard matrix representation. One easily checks that the cross-section is the set of points (t, q, p) where q lies in the equator of the sphere and p is a vector tangent to the equator at q , with all these vectors oriented the same way. The \mathbb{S}^1 -action on the cross-section is induced by rotations around the z -axis of the sphere.

For E_{triv} , a collar neighborhood of the boundary $\partial \overline{R}$ can be given by $\mathbb{S}^1 \times [0, \varepsilon) \times \mathbb{S}^1$, while for components of type E_{twist} , the form $\mathbb{R} \times [0, \varepsilon) \times \mathbb{S}^1 / \sim$ with the equivalence relation $(t, r, \varphi) \sim (t + 1, r, \varphi + \pi)$ will be used. The contact form is $dt + r d\varphi$ in both cases, and the \mathbb{S}^1 -action is $e^{i\vartheta} \cdot (t, r, \varphi) = (t, r, \varphi + \vartheta)$. The embedding of \overline{R} into the neighborhood of $M_{(\mathrm{sing})}$ is given by

$$(t, r, \varphi) \mapsto (t; (\cos \varphi, \sin \varphi, 0); r \cdot (-\sin \varphi, \cos \varphi, 0)).$$

With this embedding, the points $(t, 0, 0)$ and $(t, 0, \pi)$ in $\partial \overline{R}$ all have equal stabilizer.

This concludes the description of all singular orbits, and the proof of Lemma 8.

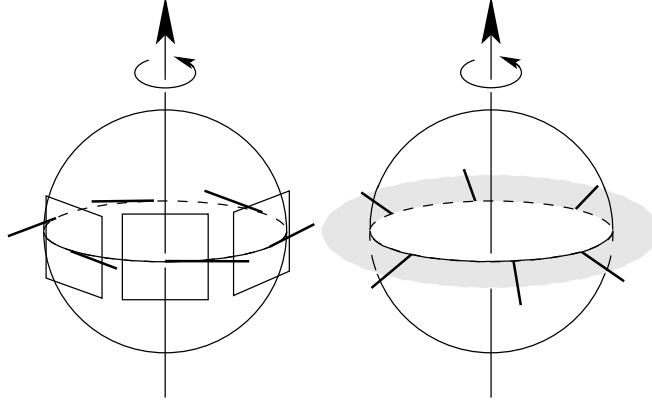


FIGURE 1. On the left the cross-section around an exceptional orbit is displayed: It consists of vectors at the equator pointing into positive direction. The picture on the right displays a model more accessible to the imagination: The cross-section sits as a ring around the equator of the sphere. Vectors pointing into the cross-section are normal to the sphere.

Example 2 (cont.). As described above, the singular orbits of \mathbb{S}^5 are composed of all points $\mathbf{x} + i\mathbf{y}$ where $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ are linearly dependent. The singular orbits are 2-spheres, and we have to decide whether the component of $\mathbb{S}^5_{(\text{sing})}$ is equal to E_{triv} or to E_{twist} . This of course is independent of the contact structure. The only points invariant under rotations around the z_3 -axis are $(0, 0, e^{i\varphi})$ with $0 \leq \varphi < 2\pi$. But since $(0, 0, 1)$ and $(0, 0, -1)$ both lie in $\text{Orb}(0, 0, 1)$, we have $\mathbb{S}^5_{(\text{sing})} \cong E_{\text{twist}}$.

Example 3 (cont.). Now we will determine the type of the singular orbits of W_k^5 . This of course does not depend on the contact structure. As we said above, a point $(z_0, z_1, z_2, z_3) \in W_k^5$ lies on a singular orbit if and only if \mathbf{x} is parallel to \mathbf{y} , where $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$. In particular, consider the points that are invariant under rotations around the z_1 -axis. They are given by $\{(e^{i\varphi}, \pm ie^{\frac{k\pm i}{2}\varphi}, 0, 0) \mid 0 \leq \varphi < 2\pi\}$. For k odd, all points lie on a single path, but for k even, there are two connected components. Hence, one obtains $(W_k^5)_{(\text{sing})} \cong E_{\text{twist}}$ for k odd, and $(W_k^5)_{(\text{sing})} \cong E_{\text{triv}}$ for k even.

So far all invariants found for $(W_k^5, \alpha_{\pm k})$, and $(W_{k'}^5, \alpha_{\pm k'})$ are equal if $k \equiv k' \pmod{2}$. But at the end of the next section, a last invariant will be computed that allows us to distinguish all of the $(W_k^5, \alpha_{\pm k})$.

2.3. Equivalence between contact $\text{SO}(3)$ -manifolds. In this section, the necessary and sufficient conditions for the existence of an $\text{SO}(3)$ -equivariant contactomorphism $\Phi : M \rightarrow M'$ between two 5-dimensional contact $\text{SO}(3)$ -manifolds (M, α) and (M', α') will be given.

If there are no singular orbits on M , then $0 \notin \mu(M)$ and the whole manifold is determined according to Theorem 1 by its cross-section. Two contact 5-manifolds with an $\text{SO}(3)$ -action without singular orbits are thus equivalent if and only if their cross-sections are. The possible cross-sections, being closed contact 3-manifold with \mathbb{S}^1 -actions, have been classified in [KT91].

On the other hand, if $0 \in \mu(M)$, then $M = M_{(\text{reg})} \cup M_{(\text{sing})}$, but there are several ways to glue both parts. The flow-out $\text{SO}(3) \cdot R \cong \text{SO}(3) \times_{\mathbb{S}^1} R$ is determined by R , but for the whole of M the problem is that $p \in \partial \overline{R}$ does not “remember” as point in the \mathbb{S}^1 -manifold \overline{R} , which stabilizer $\text{Stab}(p) \leq \text{SO}(3)$ it had in M .

The solution lies in choosing an arbitrary point $p_0 \in \partial \overline{R}$ and marking all other points p in the boundary with $\text{Stab}(p) = \text{Stab}(p_0) \leq \text{SO}(3)$. The marked points form curves in $\partial \overline{R}$. If the boundary component corresponds to E_{triv} , these curves are given by two sections to the \mathbb{S}^1 -action that are related to each other by a 180° -rotation. If the component corresponds to E_{twist} , the

marked points lie on a single curve, which intersects each \mathbb{S}^1 -orbit twice. If the singular orbits have stabilizer isomorphic to $\mathrm{O}(2)$, then the marked points form a single section.

Another way to describe the situation is the following: Gluing $M_{(\mathrm{sing})}$ onto $M_{(\mathrm{reg})}$ can be achieved by gluing R onto the cross-section in the neighborhood of $M_{(\mathrm{sing})}$. This means that one has to identify two tori. The generators of the homology in ∂R are given by an \mathbb{S}^1 -orbit and a section σ to the \mathbb{S}^1 -action in \overline{R} . The generators of the homology of $\overline{R} \cap M_{(\mathrm{sing})}$ can be described by an \mathbb{S}^1 -orbit, and by a curve of marked points as fixed above. The \mathbb{S}^1 -orbits have to coincide in both parts, and the only freedom when gluing consists in choosing the relative position of the other two homology classes.

Lemma 10. *Let (M, α) and (M', α') be two 5-dimensional contact $\mathrm{SO}(3)$ -manifolds with principal cross-sections (R, α) and (R', α') . Assume there is an \mathbb{S}^1 -contactomorphism ψ between \overline{R} and $\overline{R'}$ that maps the marked curves $\gamma_1, \dots, \gamma_n$ in $\partial \overline{R}$ onto the marked curves in $\partial \overline{R'}$, i.e. $\psi \circ \gamma_i = \gamma'_i$. Then there is an $\mathrm{SO}(3)$ -equivariant contactomorphism $\Psi : M \rightarrow M'$.*

Proof. Over the flow-out $\mathrm{SO}(3) \cdot R$ and $\mathrm{SO}(3) \cdot R'$ the claim holds. Hence if $M_{(\mathrm{sing})} = \emptyset$, then the statement is true. The problem for $\partial \overline{R} \neq \emptyset$ is that ψ extends to an $\mathrm{SO}(3)$ -homeomorphism on M , but this map is in general not smooth at the singular orbits. Hence we will need to deform ψ in a neighborhood of $\partial \overline{R}$.

Choose a component K of $M_{(\mathrm{sing})}$. The image $\psi(K)$ in $M'_{(\mathrm{sing})}$ is of the same type: If the principal stabilizer of R is isomorphic to \mathbb{Z}_2 , then every component in $M_{(\mathrm{sing})}$ and $M'_{(\mathrm{sing})}$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{RP}^2$, and if the principal stabilizer of R is trivial, then the two types of component in $M_{(\mathrm{sing})}$ and $M'_{(\mathrm{sing})}$ can be distinguished by the curves of marked points.

Now one can represent the neighborhood of K and $\psi(K)$ by the standard models described at the end of Section 2.2.2 and 2.2.3. The cross-section is either given by $(\mathbb{R} \times [0, c) \times \mathbb{S}^1 / \sim, dt + r d\varphi)$ for E_{twist} or by $(\mathbb{S}^1 \times [0, c) \times \mathbb{S}^1, dt + r d\varphi)$ for the other two types of singular orbits.

The map ψ is \mathbb{S}^1 -equivariant, thus

$$\psi(t, r, \varphi) = (T(t, r), R(t, r), \varphi + \Phi(t, r)) .$$

Furthermore it rescales the form $\alpha = dt + r d\varphi$ by a function $f(t, r) > 0$, i.e.

$$f(t, r) dt + r f(t, r) d\varphi = f\alpha = \psi^* \alpha = \left(\frac{\partial T}{\partial t} + R \cdot \frac{\partial \Phi}{\partial t} \right) dt + R d\varphi + \left(\frac{\partial T}{\partial r} + R \cdot \frac{\partial \Phi}{\partial r} \right) dr .$$

The consequences are $R(t, r) = r f(t, r)$, $\partial_t T(t, r) + r f(t, r) \cdot \partial_t \Phi(t, r) = f(t, r)$, and $\partial_r T(t, r) + r f(t, r) \cdot \partial_r \Phi(t, r) = 0$. The boundary is mapped onto the boundary, i.e. $R(t, 0) = 0$. We can assume $T(0, 0) = 0$ and $\Phi(0, 0) = 0$. Also, all of the three cases E_{triv} , E_{twist} , and $\mathbb{S}^1 \times \mathbb{RP}^2$ lead to $\Phi(t, 0) = 0$, because the γ_i are mapped onto the γ'_i .

Let $\rho_\varepsilon : \mathbb{R}^+ \rightarrow [0, 1]$ be the smooth map

$$\rho_\varepsilon(r) = \begin{cases} 0 & \text{for } r \leq \varepsilon/2 \\ N(\varepsilon) \cdot \int_{\varepsilon/2}^r \exp \frac{\varepsilon^2}{4(x-\varepsilon/2)(x-\varepsilon)} dx & \text{for } \varepsilon/2 < r < \varepsilon \\ 1 & \text{for } r \geq \varepsilon \end{cases}$$

with $N(\varepsilon)$ the reciprocal value of $\int_{\varepsilon/2}^\varepsilon \exp \frac{\varepsilon^2}{4(x-\varepsilon/2)(x-\varepsilon)} dx$. The maximum of the derivative of this function is $N(\varepsilon) \cdot \exp(-4) = N(1)e^{-4}/\varepsilon$. One can now replace the original map ψ by

$$\widehat{\psi}(t, r, \varphi) := (T(t, r), R(t, r), \varphi + \rho_\varepsilon(r) \cdot \Phi(t, r)) .$$

It is easy to check that $\widehat{\psi}$ is well-defined on the cross-section R : The relations $\psi(t+2\pi a, r, \varphi+2\pi b) = \psi(t, r, \varphi) + (2\pi a, 0, 2\pi b)$ carry over to $\widehat{\psi}$.

The map $\widehat{\psi}$ is equal to $(T(t, r), r f(t, r), \varphi)$ for points with $r \leq \varepsilon/2$ and equal to ψ for points with $r \geq \varepsilon$. It is also an \mathbb{S}^1 -diffeomorphism. The determinant of the differential $d\widehat{\psi}$ is equal to the one of $d\psi$. The injectivity and surjectivity follow easily from the same properties of ψ . For example to show that (t', r', φ') lies in the image of $\widehat{\psi}$, use that there is a (t, r, φ) with $\psi(t, r, \varphi) = (t', r', \varphi')$. Then $\widehat{\psi}(t, r, \varphi + (1 - \rho_\varepsilon(r)) \cdot \Phi(t, r)) = (t', r', \varphi')$.

There is now an $\mathrm{SO}(3)$ -diffeomorphism $\widehat{\Psi}$ on M extending $\widehat{\psi}$. Away from the singular orbits, the map $\widehat{\Psi}$ is given as in the proof of Lemma 4. In the neighborhood of $M_{(\mathrm{sing})}$ one can use the standard model for E_{triv} and E_{twist} , where the map $\widehat{\Psi}$ is given by

$$\widehat{\Psi} : (t; p, v) \mapsto (T(t, \|v\|); p, f(t, \|v\|) v) ,$$

for $p \in \mathbb{S}^2$ and for $v \in T_p^* \mathbb{S}^2$ with $\|v\| < \varepsilon/2$. If the component of $M_{(\mathrm{sing})}$ was diffeomorphic to $\mathbb{S}^1 \times \mathbb{RP}^2$ the map is given by the projectivization of $\widehat{\Psi}$ defined above. These maps clearly define $\mathrm{SO}(3)$ -equivariant diffeomorphisms in the neighborhood of a singular orbit, but one still needs to check that this definition is compatible with the map given in the proof of Lemma 4. Because both maps are $\mathrm{SO}(3)$ -equivariant, it is enough to check that these maps agree on the cross-section R . But $\widehat{\Psi}$ restricted to R gives back the map $\widehat{\psi}$. This shows that $\widehat{\Psi}$ is a globally defined map.

The map $\widehat{\Psi}$ is an $\mathrm{SO}(3)$ -diffeomorphism, but it is only a contactomorphism far away from the singular orbits. All of the $\mathrm{SO}(3)$ -invariant 1-forms in the family $\alpha_s := (1-s)\alpha + s\widehat{\Psi}^*\alpha$ on M satisfy the contact condition. This can easily be checked in a small neighborhood of the singular orbits by using the local form given above. On $M_{(\mathrm{princ})}$, one checks the contact condition along R (by choosing ε small enough) and then uses $\mathrm{SO}(3)$ -invariance. The equivariant Gray stability shows that $\widehat{\Psi}$ deforms to an $\mathrm{SO}(3)$ -contactomorphism Ψ . \square

Of course, the next question is how to find maps with the properties required in Lemma 10. For this, we need to define a last invariant for the cross-section.

Let \overline{R} be a compact oriented 3-dimensional \mathbb{S}^1 -manifold with non-empty boundary. Denote the components of $\partial \overline{R}$ by $\partial \overline{R}_j$ ($j = 1, \dots, N$) and assume that on each of the boundary components a smooth closed curve γ_j is given that intersects the \mathbb{S}^1 -orbits transversely. Orient the curves in such a way that $\dot{\gamma}_j$ followed by the infinitesimal generator Z_R of the \mathbb{S}^1 -action gives the orientation of $\partial \overline{R}_j$.

The γ_j should be of the same form as the marked points described above, i.e. if the principal stabilizer is isomorphic to \mathbb{Z}_2 , assume γ_j intersects each \mathbb{S}^1 -orbit in $\partial \overline{R}_j$ exactly once. If the principal stabilizer of R is trivial, the curves are either sections or intersect each orbit twice.

On the boundary of a small tubular neighborhood of the exceptional orbits one can define standard sections ([Orl72]), which can be extended to a global section σ of $\overline{R} \rightarrow \overline{R}/\mathbb{S}^1$. Let σ be oriented in such a way that the tangent space to the image of σ followed the generator of the \mathbb{S}^1 -action gives the positive orientation of \overline{R} .

Definition. Denote the intersection number of two oriented loops α and β in an oriented torus by $\iota(\alpha, \beta)$. If the principal stabilizer in R is trivial define the **Dehn-Euler-number** $n(R, \gamma_1, \dots, \gamma_N) \in \mathbb{Z}$ by

$$n(R, \gamma_1, \dots, \gamma_N) := 2 \sum_{j=0}^m \iota(\gamma_j, \partial \sigma) + \sum_{j=m+1}^N \iota(\gamma_j, \partial \sigma) ,$$

where we assume the first m curves to be sections to the \mathbb{S}^1 -action, and the other curves to intersect each orbit twice. Note that the first term is a sum over even numbers and the second term is a sum over odd numbers.

If the principal stabilizer is isomorphic to \mathbb{Z}_2 define the Dehn-Euler number by

$$n(R, \gamma_1, \dots, \gamma_N) := \sum_{j=1}^N \iota(\gamma_j, \partial \sigma) .$$

In this case $n(R, \gamma_1, \dots, \gamma_N)$ can be any integer.

The Dehn-Euler number is very similar to the Euler invariant for an \mathbb{S}^1 -manifold. To see that $n(R, \gamma_1, \dots, \gamma_N)$ is independent of the section chosen, assume two different sections σ_1 and σ_2 (that are homotopic to the standard sections around the exceptional orbits) are given.

There is a function $f : R/\mathbb{S}^1 \rightarrow \mathbb{S}^1$, such that $\sigma_2(p) = \sigma_1(p) \cdot f(p)$. The **rotation number** $\mathrm{rot}(f|_{\partial R_j})$ is defined as the degree of the map $f|_{\partial R_j} : \partial R_j/\mathbb{S}^1 \cong \mathbb{S}^1 \rightarrow \mathbb{S}^1$. The sum $\sum \mathrm{rot}(f|_{\partial R_j})$

over all boundary components of R vanishes, because the degree of a map $\mathbb{D}^2 \rightarrow \mathbb{S}^1$ vanishes on $\partial\mathbb{D}^2$. We can cut R/\mathbb{S}^1 open to obtain a disc, and the extra contributions from the cuts cancel out. With the equations $\iota(\gamma_j, \partial\sigma_2) - \iota(\gamma_j, \partial\sigma_1) = \mathrm{rot}(f|_{\partial R_j})$ for $j \leq k$, and $\iota(\gamma_j, \partial\sigma_2) - \iota(\gamma_j, \partial\sigma_1) = 2 \mathrm{rot}(f|_{\partial R_j})$ for $k < j \leq N$, it follows that

$$\begin{aligned} 2 \sum_{j=0}^k (\iota(\gamma_j, \partial\sigma_1) - \iota(\gamma_j, \partial\sigma_2)) + \sum_{j=k+1}^N (\iota(\gamma_j, \partial\sigma_1) - \iota(\gamma_j, \partial\sigma_2)) \\ = 2 \sum_{i=0}^k \mathrm{rot}(f|_{\partial R_i}) + \sum_{j=k+1}^N 2 \mathrm{rot}(f|_{\partial R_j}) = 0 \end{aligned}$$

Note also that the orientation of the \mathbb{S}^1 -action has no effect on $n(R, \gamma_1, \dots, \gamma_N)$. To compute $n(R, \gamma_1, \dots, \gamma_N)$ we can use again the section σ , because the standard sections around the exceptional orbits do not change with the orientation of the \mathbb{S}^1 -action. The direction of the boundary curves γ_j and the orientation of σ are inverted. But then the intersection number remains unchanged.

Remark 4. In Lemma 4, it was shown that the cross-section R (as contact \mathbb{S}^1 -manifold) is an invariant of a 5-dimensional contact manifold M . It has just been proved that the number $n(R, \gamma_1, \dots, \gamma_m)$ is also an invariant of M , because under an $\mathrm{SO}(3)$ -contactomorphism the marked curves are mapped onto each other. Below we will now finish the proof that a manifold M is completely determined by the invariants mentioned in Theorem 5 (i.e. cross-section, singular orbits and $n(R)$).

The 3-manifolds in the following lemma are cross-sections of 5-manifolds.

Lemma 11. *Let (R, α) and (R', α') be two \mathbb{S}^1 -diffeomorphic 3-dimensional contact \mathbb{S}^1 -manifolds without fixed points, but both with N boundary components. Let the orbits in the boundary be the only ones that are Legendrian. Assume further that on each of the boundary components ∂R_j and $\partial R'_i$ curves γ_j and γ'_i are specified such that for both manifolds the first k curves ($k \leq N$) are sections to the \mathbb{S}^1 -action and the other curves intersect each orbit exactly twice. Then there is an \mathbb{S}^1 -contactomorphism $\Phi : R \rightarrow R'$ such that $\Phi \circ \gamma_j = \gamma'_j$, if and only if $n(R, \gamma_1, \dots, \gamma_N) = n(R', \gamma'_1, \dots, \gamma'_N)$.*

Proof. The basic strategy is to find diffeomorphic sections with certain properties in R and R' . With these sections one can construct an \mathbb{S}^1 -diffeomorphism between the 3-manifolds that maps the boundary curves in R onto the ones in R' . Afterwards this map is deformed to obtain a contactomorphism.

By [KT91], the contact form around an exceptional orbits is locally unique up to \mathbb{S}^1 -contactomorphisms. Thus one can start the construction of Φ by taking an \mathbb{S}^1 -contactomorphism from a small neighborhood of the exceptional orbits in R to a neighborhood of the orbits of the same type in R' . Choose also, for each $j \in \{1, \dots, N-1\}$, an \mathbb{S}^1 -diffeomorphism from a neighborhood of ∂R_j to a neighborhood of $\partial R'_j$ that maps γ_j onto γ'_j .

The standard sections to the \mathbb{S}^1 -action around the exceptional orbits extend to a global section σ on $R_{(\mathrm{princ})}$. In R' , construct a section in the following way: Take σ in the neighborhood of the exceptional orbits and in the neighborhood of ∂R_j for $1 \leq j \leq N-1$ and map it with Φ to R' . Now extend the image of σ to a global section σ' on $R'_{(\mathrm{princ})}$.

By the assumptions of the lemma, we know that $n(R, \gamma_1, \dots, \gamma_N) = n(R', \gamma'_1, \dots, \gamma'_N)$, and by our construction $\iota(\sigma, \gamma_j) = \iota(\sigma', \gamma'_j)$ for all $1 \leq j \leq N-1$. It follows that the intersection numbers $\iota(\sigma, \gamma_N)$ and $\iota(\sigma', \gamma'_N)$ are also equal. Hence one can homotope σ' in such a way that its position with respect to γ'_N is the same as the one of σ with respect to γ_N .

One can map σ onto σ' and by using the \mathbb{S}^1 -action, we obtain an \mathbb{S}^1 -diffeomorphism $\Phi : R \rightarrow R'$, such that $\Phi \circ \gamma_j = \gamma'_j$ for all $j = 1, \dots, N$.

To transform the map above into a contactomorphism we need to sharpen an argument given in [Lut77] and [KT91] to avoid moving the curves on the boundaries. The neighborhoods of the boundaries are of the form $\mathbb{S}^1 \times [0, \delta) \times \mathbb{S}^1$ with coordinates (t, r, φ) , and the circle action

on the last coordinate. Assume one contact form to be $\alpha = dt + r d\varphi$ and the other one $\alpha' = g(t, r) dt + h(t, r) dr + f(t, r) d\varphi$. The orbits in the boundary are Legendrian, hence $f(t, 0) = 0$ and $\partial_t f(t, 0) = 0$. Thus the contact condition along such an orbit becomes $g(t, 0) \neq 0$, and we can divide the whole form by the function g to obtain the equivalent form $dt + h(t, r) dr + f(t, r) d\varphi$ (with new functions f and h).

Define now a map $\Psi : R \rightarrow R$ by

$$(t, r, \varphi) \mapsto (t - (1 - \rho_\varepsilon(r))rh(t, 0), r, \varphi)$$

for points with $r < \varepsilon$ and the identity otherwise. Here ρ_ε is the map defined in the proof of Lemma 10.

The map Ψ is an \mathbb{S}^1 -diffeomorphism. It is surjective, because it is the identity on the two tori $\mathbb{S}^1 \times \{0\} \times \mathbb{S}^1$ and $\mathbb{S}^1 \times \{\varepsilon\} \times \mathbb{S}^1$. The map is a local diffeomorphism because $\det(d\Psi) = 1 - r(1 - \rho_\varepsilon(r))\partial_t h(t, 0)$ does not vanish, if we choose ε small enough. Injectivity relies on a similar argument: If $\Psi(t, r, \varphi) = \Psi(t', r', \varphi')$, then clearly $\varphi = \varphi'$ and $r = r'$. Finally $t - t' = r(1 - \rho_\varepsilon(r))(h(t, 0) - h(t', 0))$. With the mean value theorem one sees that if $t \neq t'$, one has $1 = r(1 - \rho_\varepsilon(r))\partial_t h(\hat{t}, 0)$ with $\hat{t} \in (t, t')$, which is not possible if ε is chosen small enough.

For $r = 0$ the forms α and $\Psi^*\alpha'$ are equal, hence the linear interpolation $\alpha_s = (1 - s)\alpha + s\Psi^*\alpha'$ consists of \mathbb{S}^1 -invariant contact forms. To apply the Moser trick one considers the vector field X_s that is the solution to the equations

$$\iota_{X_s}\alpha_s \quad \text{and} \quad \iota_{X_s}d\alpha_s = \lambda_s\alpha_s - \dot{\alpha}_s,$$

with the function $\lambda_s := \iota_{Y_s}\dot{\alpha}_s$, where Y_s is the Reeb field of the contact form α_s . The solution X_s vanishes on ∂R , and X_s has a global flow in a small neighborhood of the boundary. Hence one has constructed an \mathbb{S}^1 -diffeomorphism between R and R' that maps the boundary curves onto each other, and respects the contact forms close to the boundaries and in the neighborhood of the exceptional orbits.

The proof is now finished by applying the Moser trick a second time, but now in the interior of the manifold. The vector field generates a global isotopy, because the two contact forms are identical close to the boundary components, and the vector field has compact support. \square

Example 2 (cont.). The Dehn-Euler number $n(R, \gamma)$ is the last invariant that needs to be computed to find $(\mathbb{S}^5, \alpha_\pm)$ in the classification scheme. The path γ can be taken to be $(e^{i\varphi}, 0, 0)$ with $0 \leq \varphi < 2\pi$, and a section in $R = \{(z_1, z_2, 0) \in \mathbb{S}^5 \mid x_1 y_2 > x_2 y_1\}$ can be found by

$$\sigma : \quad \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\} \hookrightarrow R \subset \mathbb{S}^5, \quad z \mapsto \frac{1}{\sqrt{2 + 2|z|^2}}(1 + z, z - 1, 0).$$

The boundary of σ is composed of two segments $1/\sqrt{2} \cdot (e^{i\varphi}, e^{i\varphi}, 0)$ with $\varphi \in [0, \pi]$ and $1/\sqrt{2 + 2x^2} \cdot (x + 1, x - 1, 0)$ with $x \in (-\infty, \infty)$. The boundary can be smoothed at the points where the two components meet, but this has no effect on the intersection number, because the only intersection point of $\partial\sigma$ and γ is given by $(1, 0, 0)$, and hence $n(R, \gamma) = \pm 1$. The cross-section R has opposite orientations for α_+ and α_- , thus $n_+(R, \gamma) = 1$ and $n_-(R, \gamma) = -1$.

The complete set of invariants for $(\mathbb{S}^5, \alpha_\pm)$ is: The principal stabilizer is trivial, $\mathbb{S}^5_{(\text{sing})}$ has a single component that is isomorphic to E_{twist} , the cross-section is $\mathbb{D}_{<1}^2 \times \mathbb{S}^1$, and the Dehn-Euler number $n(R)$ equals ± 1 .

Example 3 (cont.). Above, we already saw that the cross-section of any W_k^5 is \mathbb{S}^1 -diffeomorphic to $\mathbb{D}_{<1}^2 \times \mathbb{S}^1$ and $(W_k^5)_{(\text{sing})}$ is isomorphic to E_{triv} for k even and E_{twist} for k odd.

Now, we will compute $n(R, \gamma)$ for (W_k^5, α_k) and (W_k^5, α_{-k}) . The curve $\gamma(\varphi)$ is given by $(e^{i\varphi}, +ie^{\frac{k}{2}i\varphi}, 0, 0)$ with $\varphi \in [0, 2\pi]$ for k even and with $\varphi \in [0, 4\pi]$ for k odd.

Set $r_0 = |z_0|$ and $A = \sqrt{2 - r_0^2 + \sqrt{(2 - r_0^2)^2 - r_0^{2k}}}$. The map below is a section of \overline{R}

$$\sigma : \quad \mathbb{D}^2 \hookrightarrow \overline{R}, \quad z_0 \mapsto \left(z_0, \frac{iz_0^k}{2A} + \frac{iA}{2}, -\frac{z_0^k}{2A} + \frac{A}{2}, 0 \right).$$

The restriction of σ to $\partial\overline{R}$ is $\sigma(\varphi) = (e^{i\varphi}, \frac{i}{2}(1 + e^{ik\varphi}), \frac{1}{2}(1 - e^{ik\varphi}), 0)$.

The intersection of γ and $\partial\sigma$ is given by the equations $2e^{i\frac{k}{2}\varphi} = 1 + e^{ik\varphi}$ and $1 - e^{ik\varphi} = 0$, and hence $k\varphi = 4\pi n$ with $n \in \mathbb{Z}$. For $k = 0$, every point of $\partial\sigma$ lies in the curve of marked points, but by shifting the section a bit with the \mathbb{S}^1 -action, one obtains $n(R, \gamma) = 0$. For k even, the curve γ is parametrized by $\varphi \in [0, 2\pi)$, and there are $k/2$ intersection points, for k odd, the curve γ closes for $\varphi \in [0, 4\pi)$, and there are k intersection points.

The calculations so far did not depend on the contact form, but one can check that R has different orientations for α_{+k} and α_{-k} . This changes the orientation of $\partial\sigma$ and γ , but also of ∂R , and hence for (W_k^5, α_k) , we have $n(R, \gamma) = k$, and for (W_k^5, α_{-k}) , we have $n(R, \gamma) = -k$.

The complete set of invariants for $(W_k^5, \alpha_{\pm k})$ is: The principal stabilizer is trivial, $(W_k^5)_{(\mathrm{sing})}$ is isomorphic to E_{twist} for k odd and to E_{triv} for k even, the cross-section is $\mathbb{D}_{<1}^2 \times \mathbb{S}^1$, and $n(R) = \pm k$. In particular it follows that the 5-sphere (\mathbb{S}^5, α_+) in Example 2 is equivalent to (W_1^5, α_{+1}) , and (\mathbb{S}^5, α_-) is equivalent to (W_1^5, α_{-1}) .

Note also that every 5-dimensional simply connected contact $\mathrm{SO}(3)$ -manifolds with singular orbits is $\mathrm{SO}(3)$ -contactomorphic to one of the Brieskorn examples $(W_k^5, \alpha_{\pm k})$. The reason is that the orbit space $M/\mathrm{SO}(3)$ of M has to be simply connected ([Bre72]), and must have boundary. Hence $M/\mathrm{SO}(3)$ is a 2-disc, and $M_{(\mathrm{sing})}$ has a single component. From this it follows that the cross-section is isomorphic to $\mathbb{D}_{<1}^2 \times \mathbb{S}^1$. If the principal stabilizer was isomorphic to \mathbb{Z}_2 , then it is easy to show by applying the Theorem of Seifert-van Kampen that $\pi_1(M) \cong \mathbb{Z}_2$. Thus, the principal stabilizer has to be trivial, and all cases are covered by the W_k^5 .

2.4. Construction of 5-manifolds. In this section, we will construct a manifold M for each of the possible combination of invariants given in Theorem 5.

2.4.1. $M_{(\mathrm{sing})} = \emptyset$. The classification given in [KT91] shows that there is an \mathbb{S}^1 -invariant contact structure without Legendrian orbits on any closed 3-dimensional contact \mathbb{S}^1 -manifolds R that does not have special exceptional orbits or fixed points, and whose (orbifold) Euler number does not vanish. The 5-manifold M is then given by $M \cong \mathrm{SO}(3) \times_{\mathbb{S}^1} R$, where the circle on R acts with k -fold speed to get the desired stabilizer on M .

On the other hand, it follows from Lemma 7 that $0 \notin \mu(M)$, and thus R cannot have Legendrian orbits. It is also clear that R cannot have fixed points.

2.4.2. $M_{(\mathrm{sing})} \neq \emptyset$ and trivial principal stabilizer. Let \overline{R} be any 3-dimensional \mathbb{S}^1 -manifold without fixed points and without special exceptional orbits, but with non-empty boundary ∂R . By the requirement that only the \mathbb{S}^1 -orbits on the boundary are Legendrian, the contact structure on \overline{R} is uniquely determined ([KT91]).

Over the interior of \overline{R} , the 5-manifold $M^* = \mathrm{SO}(3) \times_{\mathbb{S}^1} (R - \partial R)$ is a contact $\mathrm{SO}(3)$ -manifold. Now one has to glue in the singular orbits, in such a way as to get the chosen combination of components of type E_{triv} and E_{twist} and the Dehn-Euler number $n(R)$. First we will show how to glue in the standard model for E_{triv} ; for this, we need to have a standard form for a neighborhood of ∂R .

Let σ be any section in \overline{R} that is compatible with the standard sections around the exceptional orbits. In [Lut77] it has been shown that any contact form around ∂R is equivalent to a standard form: Denote the coordinates of a collar $\mathbb{S}^1 \times [0, \varepsilon) \times \mathbb{S}^1$ around a boundary component by (t, r, φ) and let the \mathbb{S}^1 -action be $e^{i\vartheta} \cdot (t, r, \varphi) = (t, r, \varphi + \vartheta)$. Every invariant contact form is up to an \mathbb{S}^1 -contactomorphism equal to $dt + r d\varphi$. In general the section σ will not be of the form $\sigma(e^{it}, r) = (e^{it}, r, 1)$ in the collar though, but it is not very difficult to arrange the model neighborhood in this way. Let $[t]$ and $[\varphi] \in H_1(M, \mathbb{Z})$ be the classes given by $\mathbb{S}^1 \times \{0\} \times \{1\}$ and $\{1\} \times \{0\} \times \mathbb{S}^1$. The section σ represents an element $[t] + a[\varphi]$, and there is a linear map $A \in \mathrm{SL}(2, \mathbb{Z})$ that induces an \mathbb{S}^1 -diffeomorphism, such that σ represents $[t]$ in the new coordinates. The contact form becomes $(1 + ar) dt + r d\varphi$, which after dividing by $1 + ar$ and rescaling in the r -direction can be transformed back into $dt + r d\varphi$. Now by deforming σ , one obtains a collar for the boundary where the action, the contact form and the section are all in standard form.

The standard way of gluing is to consider $\mathbb{S}^1 \times T^*\mathbb{S}^2$ with $\mathrm{SO}(3)$ -action on the second factor and with the contact form $dt + \lambda_{\mathrm{can}}$. The cross-section of $\mathbb{S}^1 \times T^*\mathbb{S}^2$ looks exactly like the neighborhood of the boundary components of \overline{R} , which allows us to identify both. Since the cross-section

determines the 5-manifold lying over it, this gives a gluing of $\mathbb{S}^1 \times T^*\mathbb{S}^2$ to M^* . In the boundary, the section σ and the curve of marked points are identical, but one can push σ a bit along the \mathbb{S}^1 -action to avoid having any intersection points. Thus the contribution of this gluing to $n(R)$ is zero.

To construct a general M , i.e. an M with $n(R) \neq 0$ or with $E_{\text{twist}} \subset M_{(\text{sing})}$, we need to change the construction.

Assume first that we want to glue in a component of type E_{triv} , which adds $2c$ to the Dehn-Euler number. The neighborhood of ∂R was chosen above to be $\mathbb{S}^1 \times [0, \varepsilon) \times \mathbb{S}^1$ with contact form $dt + r d\varphi$ and with a section σ of the form $\sigma(e^{it}, r) = (e^{it}, r, 1)$. The matrix

$$A = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$$

induces a diffeomorphism, which can be isotoped as above to obtain a new model for the neighborhood of ∂R , where σ represents the homology class $[t] + c[\varphi]$, and where the contact form is still in standard form. Gluing E_{triv} along the cross-section \bar{R} works again without any problem. The intersection number between the section σ and the curve of marked points gives now c .

To glue in a component of type E_{twist} , recall that the cross-section around E_{twist} could be described by $\mathbb{R} \times [0, \varepsilon) \times \mathbb{S}^1 / \sim$ with the equivalence relation $(t, r, e^{i\varphi}) \sim (t+1, r, e^{i(\varphi+\pi)})$ and contact form $\alpha = dt + r d\varphi$. The curve of marked points was given by $\{(t, 0, 1)\}$ and $\{(t, 0, -1)\}$. There is now a diffeomorphism $\Phi : \mathbb{S}^1 \times [0, \varepsilon) \times \mathbb{S}^1 \rightarrow \mathbb{R} \times [0, \varepsilon) \times \mathbb{S}^1 / \sim$, $(e^{2\pi it}, r, e^{i\varphi}) \mapsto (t, r, e^{i(\varphi+\pi t/2)})$. The curve of marked points pulls back to $\{(e^{2\pi it}, 0, e^{-\pi it})\}$, and $\Phi^*\alpha = (1 + \pi r/2) dt + r d\varphi$, which can be isotoped into standard form. The model for the cross-section close to E_{twist} and close to ∂R looks identical, and it is possible to glue both parts. The Dehn-Euler number $n(R)$ can be arranged in the desired way as above.

2.4.3. $M_{(\text{sing})} \neq \emptyset$ and principal stabilizer is \mathbb{Z}_2 . If the principal stabilizer is isomorphic to \mathbb{Z}_2 , then all components of $M_{(\text{sing})}$ are equivalent to $\mathbb{S}^1 \times \mathbb{RP}^2$. The gluing occurs completely analogous to the way it was done above: Choose identical charts for a neighborhood of ∂R , and for the cross-section around $M_{(\text{sing})}$, and glue along these.

2.5. Relation between the Dehn-Euler number and generalized Dehn twists. In this section we want to show that the Dehn-Euler number $n(R)$ counts the number of Dehn twists needed to glue in the singular orbits.

A **k -fold Dehn twist** τ_k on $T^*\mathbb{S}^2$ can be constructed in the following way. Write a point in $T^*\mathbb{S}^2$ as $(\mathbf{q}, \mathbf{p}) \in \mathbb{R}^3 \times \mathbb{R}^3$ with $|\mathbf{q}| = 1$ and $\mathbf{q} \perp \mathbf{p}$.

If one chooses in the map

$$\tau_k(\mathbf{q}, \mathbf{p}) = \left(\mathbf{q} \cdot \cos f(|\mathbf{p}|) + \frac{\mathbf{p}}{|\mathbf{p}|} \cdot \sin f(|\mathbf{p}|), \mathbf{p} \cdot \cos f(|\mathbf{p}|) - |\mathbf{p}| \cdot \mathbf{q} \cdot \sin f(|\mathbf{p}|) \right)$$

the function f to be $f(r) = r$, then τ_k is just the standard geodesic flow. Instead, we will use $f(r) = \pi k(1 + \rho_\varepsilon(r))$ with ρ_ε as defined in the proof of Lemma 10. The map is $\text{SO}(3)$ -equivariant, and for small \mathbf{p} , the map is $\tau_k \equiv (-1)^k \text{id}$, while for large \mathbf{p} , it is $\tau_k \equiv \text{id}$.

The canonical 1-form transforms like

$$\tau_k^* \lambda_{\text{can}} = \lambda_{\text{can}} + |\mathbf{p}| d(f(|\mathbf{p}|)) .$$

This shows that τ_k would be a symplectomorphism of $(T^*\mathbb{S}^2, d\lambda_{\text{can}})$, but not a contactomorphism of $dt + \lambda_{\text{can}}$ on $\mathbb{R} \times T^*\mathbb{S}^2$. It is known ([Sei98]) that τ_{2n} is isotopic to id and τ_{2n+1} is isotopic to τ_1 (both in the space of diffeomorphisms with compact support).

The mapping torus

$$\mathbb{R} \times T^*\mathbb{S}^2 / \sim, \text{ where } (t; \mathbf{q}, \mathbf{p}) \sim (t+1; \tau_k(\mathbf{q}, \mathbf{p}))$$

is then diffeomorphic to V^*E_{triv} for k even and to V^*E_{twist} for k odd.

The 1-form

$$\alpha = dt + \lambda_{\text{can}} - t|\mathbf{p}| df$$

on $\mathbb{R} \times T^*\mathbb{S}^2$ is invariant under the equivalence relation, and thus projects down onto the mapping torus. The contact condition for α gives

$$0 \neq \alpha \wedge (d\alpha)^2 = dt \wedge (d\lambda_{\text{can}})^2 - 2|\mathbf{p}| \lambda_{\text{can}} \wedge dt \wedge df \wedge d\lambda_{\text{can}} = (1 - 2|\mathbf{p}|^2 f') dt \wedge d\lambda_{\text{can}}^2.$$

Because $\max f' = c/\varepsilon$, by choosing ε small enough we can assure that $1 - 2|\mathbf{p}|^2 f' \neq 0$, and α is then an $SO(3)$ -invariant contact form.

In fact, because the map τ_k used to construct the mapping torus is the identity far away from the zero-section and because the term $t|\mathbf{p}|df$ in α disappears there, it is possible to cut out a component of $M_{(\text{sing})}$ and glue in $\mathbb{R} \times T^*\mathbb{S}^2/\sim$ to obtain a new contact $SO(3)$ -manifold.

It only remains to see what effect this has on the integer $n(R)$.

The cross-section R in $\mathbb{R} \times T^*\mathbb{S}^2$ is equal to the one for the standard contact form

$$R = \left\{ (t; (x, y, 0), (ry, -rx, 0)) \right\} / \sim,$$

because the last term in α does not change the moment map ($\iota_{X_M} df = \mathcal{L}_{X_M} f$, but f only changes in radial direction).

To compute the local contribution to $n(R)$, notice that the section $\sigma(t, r) = (t; (1, 0, 0), (0, -r, 0))$ on $\mathbb{R} \times T^*\mathbb{S}^2$ does not descend to a continuous section on the mapping torus. Instead one could replace σ by

$$\sigma(t, r) = (t; (\cos tf(r), -\sin tf(r), 0), (-r \cdot \sin tf(r), -r \cdot \cos tf(r), 0)).$$

Since σ remains unchanged far away from the singular orbits, it extends to the unmodified section, and it is easy to check that σ induces a continuous section on $\mathbb{R} \times T^*\mathbb{S}^2/\sim$.

The intersections of σ with the curve of marked points is given by $(\cos tf(0), -\sin tf(0), 0) = (\pm 1, 0, 0)$, i.e. $\cos \pi kt = \pm 1$ and $\sin \pi kt = 0$, and then $kt \in \mathbb{Z}$. There are k points on ∂R , where σ intersects the marked set of points.

If k is odd, the boundary corresponds to E_{twist} . Then there is only a single curve of marked points and the contribution of this boundary to $n(R)$ is k . If k is even, then there are two disjoint curves of marked points, and there are only $k/2$ intersection points with the first one. But since for singular orbits of type E_{triv} this number is multiplied by 2, the contribution to $n(R)$ is again k .

Thus the Dehn-Euler number $n(R)$ counts the number of Dehn twists applied at $M_{(\text{sing})}$.

All constructions on $\mathbb{S}^1 \times \mathbb{S}^2$ in this section are \mathbb{Z}_2 -equivariant, and this allows us to build manifolds with principal stabilizer \mathbb{Z}_2 and arbitrary $n(R)$.

REFERENCES

- [Aud91] M. Audin, *The topology of torus actions on symplectic manifolds*, Progress in Mathematics, 93. Basel : Birkhäuser Verlag, 1991.
- [Bre72] G.E. Bredon, *Introduction to compact transformation groups*, Academic Press, New York, 1972, Pure and Applied Mathematics, Vol. 46.
- [DK00] J.J. Duistermaat and J.A.C. Kolk, *Lie groups*, Universitext. Berlin: Springer, 2000.
- [HM68] F. Hirzebruch and K. H. Mayer, *O(n)-Mannigfaltigkeiten, exotische Sphären und Singularitäten*, Lecture Notes in Mathematics, No. 57, Springer-Verlag, Berlin, 1968.
- [Igl91] P. Iglesias, *Les $SO(3)$ -variétés symplectiques et leur classification en dimension 4. (Symplectic $SO(3)$ -manifolds and their classification in dimension 4)*, Bull. Soc. Math. Fr. **119** (1991), no. 3, 371–396.
- [KT91] Y. Kamishima and T. Tsuboi, *CR-structures on Seifert manifolds*, Invent. Math. **104** (1991), no. 1, 149–163.
- [LM76] R. Lutz and C. Meckert, *Structures de contact sur certaines sphères exotiques*, C. R. Acad. Sci. Paris Sér. A-B **282** (1976), no. 11, A591–A593.
- [LMTW98] E. Lerman, E. Meinrenken, S. Tolman, and C. Woodward, *Non-Abelian convexity by symplectic cuts*, Topology **37** (1998), no. 2, 245–259.
- [Lut77] R. Lutz, *Structures de contact sur les fibrés principaux en cercles de dimension trois*, Ann. Inst. Fourier (Grenoble) **27** (1977), no. 3, ix, 1–15.
- [LW01] E. Lerman and C. Willett, *The topological structure of contact and symplectic quotients*, Int. Math. Res. Not. **2001** (2001), no. 1, 33–52.
- [Orl72] P. Orlik, *Seifert manifolds*, Lecture Notes in Mathematics. 291. Berlin: Springer-Verlag. VIII, 1972.
- [Sei98] P. Seidel, *Symplectic automorphisms of $T^*\mathbb{S}^2$* , arXiv **math.DG/9803084** (1998).
- [vKN] Otto van Koert and Klaus Niederkrüger, *Open book decompositions for contact structures on Brieskorn manifolds*, arXiv **math.SG/0405029**.
- [Wil02] C. Willett, *Contact reduction*, Trans. Amer. Math. Soc. **354** (2002), no. 10, 4245–4260 (electronic).

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